INTERSCHOOL MATHEMATICAL COMPETITION 1985

PART A

Saturday, 29 June 1985 1000 – 1100

Attempt as many questions as you can. Circle your answers on the Answer Sheet provided.

Each question carries 5 marks

1. The largest positive integer which divides $n^5 - n$ for all positive integers n is:

   (a) 1, (b) 15, (c) 30, (d) 60, (e) none of the preceding

2. Let r, s be roots of $ax^2 + bx + c = 0$. Then the equation whose roots are $ar + b$ and $as + b$ is:
   (a) $x^2 - bx - ac = 0$
   (b) $x^2 - bx + ac = 0$
   (c) $x^2 + 3bx + ca + 2b^2 = 0$
   (d) $x^2 + 3bx - ca + 2b^2 = 0$
   (e) none of the preceding.

3. The remainder when $x^{120} + 1$ is divided by $(x + 1)^2$ is:
   (a) 2, (b) $x + 3$, (c) $-x + 1$, (d) $2x + 4$, (e) none of the preceding

4. Let m, n be integers with $1 < m \leq n$. We define:

$$f(m, n) = (1 - \frac{1}{m})(1 - \frac{1}{m+1})(1 - \frac{1}{m+2}) \dots (1 - \frac{1}{n}).$$

   Then $f(2, n) + f(3, n) + \dots + f(n, n)$ is equal to:
   (a) $1/n$, (b) $n/2$, (c) $(n-1)/2$, (d) $n(n-1)$,
   (e) none of the preceding

5. A line segment is divided into three random parts. Then the probability that these three parts form the sides of a triangle is:
   (a) 1/4 (b) 1/3 (c) 1/2 (d) 1/5 (e) none of the preceding

6. Let $695 = a_1 + a_2.2! + a_3.3! + \dots + a_n.n!$, where $a_1, a_2, \dots, a_n$ are integers and $0 \leq a_k \leq k$ for each k. Then a_4 is equal to:
   (a) 0, (b) 1, (c) 2, (d) 3, (e) none of the preceding.

7. Let S be the solution set of the simultaneous equations $(x-1)^2 + (y+2)^2 + (z-5)^2 = 64$ and $(x+3)^2 + (y-1)^2 + (z+7)^2 = 25$, where x, y and z are real numbers. Then,

(a) S is an empty set.
(b) S is a singleton.
(c) S is a finite set with more than one element.
(d) S represents a straight line.
(e) none of the preceding.

8. The value of $\int_{\pi/3}^{2\pi/3} \frac{x}{\sin x}\, dx$ is equal to:

(a) $\frac{\pi}{2}\ell n3$ (b) $\frac{\pi}{3}\ell n3$, (c) $\frac{\pi}{4}\ell n3$, (d) $\frac{\pi}{6}\ell n3$, (e) none of the preceding

9. Given that $2x + y = 12$, the maximum value of $\log_4 x + \log_2 y$ is:
(a) 9/2, (b) 4, (c) 7/2, (d) 7/3, (e) none of the preceding

10. Let A be a 3 by 3 determinant such that all the entries of A are between -1 and 1 inclusive of -1 and 1. Then the maximum value of A is:
(a) 4, (b) 4.5, (c) 5, (d) 5.5, (e) none of the preceding.

INTERSCHOOL MATHEMATICAL COMPETITION 1985

PART B

Saturday, 29 June 1985 1100 – 1300

Attempt as many questions as you can.
Each question carries 25 marks.

1. Let ABC be a triangle with $\angle A = 4\angle C$ and $\angle B = 2\angle C$. Show that $(\overline{BC} + \overline{CA})\overline{AB} = \overline{BC}\,\overline{CA}$.

2. X, Y and Z are integers such that $X^3 + 3Y^3 + 9Z^3 = 0$. Prove that $X = Y = Z = 0$.

3. A real polynomial $p(x) = ax^2 + bx + c$ ($a \geq 0$ and $b \geq 0$) is such that $|p(x)| \leq 1$ for $|x| \leq 1$. Let $q(x) = cx^2 + bx + a$. Show that $|q(x)| \leq 2$ for $|x| \leq 1$.

4. Prove that $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ for $n > 1$ is never an integer.

5. Denote by (x,y) the greatest common divisor of two positive integers x and y. Let a and b be two positive integers such that $(a, b) = 1$ and let $p \geq 3$ be a prime. Denote

$$\alpha = \left(a+b, \frac{a^p + b^p}{a+b}\right).$$

Show that

(i) $(\alpha, a) = 1$

(ii) $\alpha = 1$ or $\alpha = p$.

6. Let n and r be integers with $0 \leq r \leq n$.

(i) Show that

$$\binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \ldots + \binom{r}{r} = \binom{n+1}{r+1}.$$

Assume $r \geq 1$ and let $\left\{A_1, A_2, \ldots, A_{\binom{n}{r}}\right\}$ be the collection of r-element subsets of the set $S = \{1, 2, \ldots, n\}$. For each $i = 1, 2, \ldots, \binom{n}{r}$, denote by m_i the smallest number in A. Show that

(ii) $1 \leq m_i \leq n-r+1$;

(iii) $m_1 + m_2 + \ldots + m_{\binom{n}{r}} = \sum_{m=1}^{n-r+1} \binom{n-m}{r-1} m$; and

(iv) the arithmetic mean of $m_1, m_2, \ldots, m_{\binom{n}{r}}$ is $(n+1) / (r+1)$.

INTERSCHOOL MATHEMATICAL COMPETITION 1985

Solutions

Part A

1. We have $n^5 - n = n(n-1)(n+1)(n^2+1)$. As $n-1$, n and $n+1$ are three consecutive integers, we see that $(n-1)n(n+1)$ is divisible by both 2 and 3 and hence by 6. Moreover if the unit digit of n is 0, 1, 4, 5, 6, 8 or 9, then $(n-1)n(n+1)$ will also be divisible by 5 and hence by 30; while if the unit digit of n is 2, 3, or 7, then n^2+1 will be divisible by 5. Hence, in any case n^5-n is always divisible by 30. As n^5-n is equal to 30 when $n = 2$, the correct answer is (c).

2. As r, s are roots of $ax^2+bx+c=0$, we have $r+s = -b/a$ and $rs = c/a$. Now, $(ar+b) + (as+b) = a(r+s)+2b = b$ and $(ar+b)(as+b) = a^2rs+ab(r+s)+b^2 = ac-b^2+b^2 = c$. So the equation whose roots are $ar+b$ and $as+b$ is: $x^2-bx+ac=0$. Hence the correct answer is (b).

3. Let $x^{120} + 1 = (x+1)^2\,Q(x) + ax + b$. Put $x = -1$, we have

$$-a + b = 2 \text{———} \quad (\mathrm{I})$$

Differentiating the above equation with respect to x, we have:

$$120x^{119} = 2(x+1)\,Q(x) + (x+1)^2\,Q'(x) + a.$$

Put $x = -1$, we obtain:

$$a = -120 \text{————} (\mathrm{II})$$

From (I) and (II), we obtain $b = -118$. So the remainder is

$$-120x - 118.$$

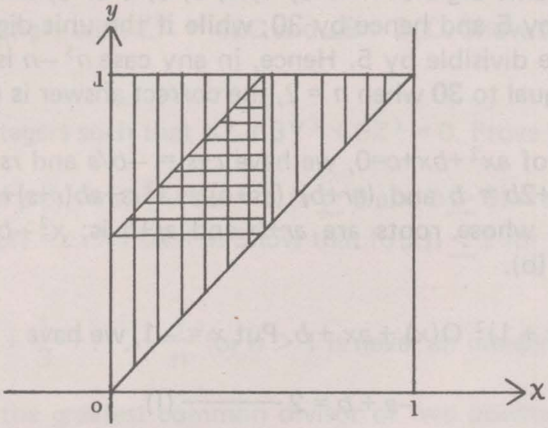
The correct answer is (e).

4. We have $f(m, n) = [(m-1)m(m+1)\ldots(n-1)]/[m(m+1)(m+2)\ldots n] = (m-1)/n$. Thus $f(2, n) + f(3, n) + \ldots + f(n, n) = [1 + 2 + \ldots + (n-1)]/n = (n-1)/2$. Therefore the correct answer is (c).

5. Without loss of generality, we may assume that the line segment is the unit interval on the real line from 0 to 1. To divide the line segment into three random parts, we need only to choose two random numbers $0 \leq x \leq y \leq 1$. All such pairs (x, y) form a triangle with area 1/2 in the xy-plane as shaded in the following figure. The three parts will form the sides of a triangle if the inequalities as given below hold:

(i) $x + (y-x) > 1 - y$ i.e. $y > 1/2$
(ii) $x + (1 - y) > y - x$ i.e. $y - x < 1/2$
(iii) $(y - x) + (1 - y) > x$ i.e. $x < 1/2$.

All pairs (x, y) satisfying (i), (ii) and (iii) above form a triangle with area 1/8, as indicated in the doubly shaded region in the figure. Hence, the required probability is [1/8] / [1/2] = 1/4. The correct answer is (a).



6. The largest n with $n! \leq 695$ is 5. As 5 x 5! = 600 < 695, we have $a_5 = 5$. Now 695–600 = 95 and the largest n with $n! \leq 95$ is 4. As 3 x 4! = 72 < 95, we have $a_4 = 3$. So the correct answer is (d).

7. In R^3, the equation $(x - 1)^2 + (y + 2)^2 + (z - 5)^2 = 64$ represents a sphere with centre at the point A(1, –2, 5) and radius equal to 8 units. The equation $(x + 3)^2 + (y-1)^2 + (z + 7)^2 = 25$ also represents a sphere with centre at the point B(–3, 1, –7) and radius equal to 5 units. Now, AB = 13 = 8 + 5, from which we conclude that S must be a singleton. Hence the correct answer is (b).

8. Let $y = \pi - x$. Then we have

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} (x/\sin x)\, dx = -\int_{2\pi/3}^{\pi/3} (\pi - y)/\sin(\pi - y)\, dy$$

$$= \int_{\pi/3}^{\frac{2\pi}{3}} (\pi - y) \sin y \ dy$$

$$= \int_{\pi/3}^{\frac{2\pi}{3}} \pi/\sin y\, dy - \int_{\pi/3}^{\frac{2\pi}{3}} y/\sin y\, dy$$

$$= \left[\ell n(\operatorname{cosec} y - \cot y)\right]_{\pi/3}^{\frac{2\pi}{3}} - \int_{\pi/3}^{\frac{2\pi}{3}} y/\sin y\, dy.$$

Hence, $\displaystyle\int_{\pi/3}^{\frac{2\pi}{3}} x/\sin x\, dx = \pi/2\,\left[\ell n(\operatorname{cosec} y - \cot y)\right]_{\pi/3}^{\frac{2\pi}{3}}$

$$= \frac{\pi}{2}\,\ell n 3.$$

So the correct answer is (a).

9. We have $y = 12 - 2x = 2(6 - x)$, $y > 0$ and $x > 0$. Hence $0 < x < 6$.

Moreover,
$$\begin{aligned}\log_4 x + \log_2 y &= \log_2 x/\log_2 4 + \log_2 2\,(6-x)\\ &= (1/2)\log_2 x + 1 + \log_2 (6-x)\\ &= (1/2)(2 + \log_2 x + 2\log_2 (6-x))\\ &= (1/2)(2 + \log_2 x\,(6-x)^2).\end{aligned}$$

Let $f(x) = x(6-x)^2$. Then $f'(x) = 3(x-2)(x-6)$ and $f''(x) = 6(x-4)$. We see that $f'(x) = 0$ iff $x = 2$ for $0 < x < 6$. Also, $f''(2) = -12 < 0$. So $f(x)$ has a maximum value when $x = 2$. Also when $x = 2$, $\log_4 x + \log_2 y = (1/2)(2 + \log_2 32) = 7/2$. The correct answer is therefore (c).

10. Let (a_{ij}) be the 3 x 3 matrices with entries a_{ij} and let A_{ij} be the determinant by deleting the ith row and the jth column from A. Then

$$A = \sum_{j=1}^{3} (-1)^{i+j} a_{ij} A_{ij} \text{ for } i = 1, 2, 3.$$

It is easy to see that we can change $|a_{ij}|$ to 1 without decreasing the value of A. Hence A can achieve its maximum value when $|a_{ij}| = 1$ for all i, j. Thus we may assume that $|a_{ij}| = 1$ for all i, j.

Next, we note that A is the volume of the parallelepiped formed by the vectors (a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}) and (a_{31}, a_{32}, a_{33}). Hence A $\leq$

$(\sqrt{3})^3 = \sqrt{27} < 6.$

Finally, we see that A is the sum of six terms which are either + 1 or −1. Since $A < 6$, at least one of these six terms must be −1. Hence $A \leq 4$. As

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 4$$

we conclude that the maximum value of A is 4. The correct answer is therefore (a).

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Solutions

Part B

1. We have $\angle A = 4\angle C$, $\angle B = 2\angle c$ and $\angle A + \angle B + \angle C = \pi$. Hence $\angle C = \pi/7$, $\angle B = 2\angle C$ and $\angle A = 4\angle C$ From this, we have: $BC = k\sin(4\angle C)$, $CA = k\sin(2\angle C)$ and $AB = k\sin(\angle C)$ for some constant k. Therefore we need only to show now that $(\sin(4\angle C) + \sin(2\angle C))\sin(\angle C) = \sin(4\angle C)\sin(2\angle C)$. Indeed, we have:
$$\begin{aligned}(\sin(4\angle C) + \sin(2\angle C)\sin(\angle C) &= (2\sin(3\angle C)\cos(\angle C))\sin(\angle C)\\ &= \sin(3\angle C)\sin(2\angle C)\\ &= \sin(\pi - 3\angle C)\sin(2\angle C)\\ &= \sin(4\angle C)\sin(2\angle C), \text{ as required.}\end{aligned}$$

2. Suppose to the contrary that there exist integers X, Y, and Z not all zero satisfying the given equation. Without loss of generality, we may assume the greatest common divisor of X, Y and Z is 1. As $X^3 + 3Y^3 + 9Z^3 = 0$, we see that 3 divides X, say $X = 3A$ for some integer A. By substitution, we get $27A^3 + 3Y^3 + 9Z^3 = 0$. Dividing this equation throughout by 3, we get $Y^3 + 3Z^3 + 9A^3 = 0$. Therefore, 3 divides Y, say $Y = 3B$. Again, by substitution, we have $27B^3 + 3Z^3 + 9A^3 = 0$, i.e. $Z^3 + 3A^3 + 9B^3 = 0$. Hence, we have 3 divides Z. This however contradicts the fact that X, Y and Z have greatest common divisor 1. We thus conclude that $X = Y = Z = 0$, as required.

3. Substituting $x = 1$, 0 and -1 respectively into $p(x)$, we get

   $|a + b + c| \leq 1$

   $|c| \leq 1$ and

   $|a - b + c| \leq 1$.

   Therefore, $|a + b| = |a + b + c - c| \leq |(a + b + c)| + |c| \leq 2$ and $|a - b| = |a - b + c - c| \leq |a - b + c| + |c| \leq 2$. Now let x be any real number with $|x| \leq 1$. If $c \geq 0$ then we have: $0 \leq cx^2 \leq c$ and $-b \leq bx \leq b$. Thus, $-2 \leq a-b = 0 + (-b) + a \leq q(x) = cx^2 + bx + a \leq c + b + a \leq 1$. Hence $|q(x)| \leq 2$ for $|x| \leq 1$, as required. On the other hand, if $c < 0$, we have $c \leq cx^2 \leq 0$ and $-b \leq bx \leq b$. Then, $-1 \leq c - b + a \leq q(x) = cx^2 + bx + a \leq 0 + b + a \leq 2$. This again gives $|q(x)|$

   ≤ 2, which completes the proof.

4. Assume $\sum_{i=1}^{n} 1/i \in Z$. We may assume that $n > 2$. Let the least common multiple of 1, 2, . . . , n be a. There is an r in Z ($r>0$) with $2^r \leq n < 2^{r+1}$. Let $a = 2^r 3^t 5^s \ldots$ be the prime decompostition of a. Since $n > 2$, we have $a/2 \in Z$. We

then have: $(\frac{1}{2}a)/2^r \notin Z$ by definition of r, and

$$(\frac{1}{2}a)/j \in Z \text{ for } 1 \leq j \leq n, j \neq 2^r \text{ ----(*)}$$

But $a/2 \in Z$, and by assumption $\sum_{j=1}^{n} 1/i \in Z$, thus $\frac{1}{2}a\,(\sum_{i=1}^{n} 1/i) \in Z$. On the other hand, by (*), we find that $\frac{1}{2}a\,(\sum_{j=1}^{n} 1/j) \notin Z$. This is a contadiction.

5. We have $a + b = \alpha t$ and $(a^p + b^p)/(a + b) = \alpha s$ for some positive integers t and s. Therefore $\alpha^2 ts = a^p + b^p = a^p + (\alpha t - a)^p = \alpha^p t^p - pa\alpha^{p-1}t^{p-1} + \ldots + p\alpha t a^{p-1}$. Hence $\alpha s = \alpha^{p-1}t^{p-1} - pa\alpha^{p-2}t^{p-2} + \ldots + pa^{p-1}$. From this, we see that $\alpha |\, pa^{p-1}$. We shall show that $(\alpha, a) = 1$. Suppose to the contrary that $(\alpha, a) = k > 1$. Let q be a prime factor of k. Clearly, $q|k$, $q|\alpha$ and $q|a$. Since $\alpha |\, a + b$, $q |\, a + b$ and so $q|b$. But then $(a, b) \neq 1$, a contradiction. Thus $(\alpha, a) = 1$, as required.
Now, since $\alpha | pa^{p-1}$ and $(\alpha, a) = 1$, it follows that $\alpha |\, p$ and so $\alpha = 1$ or $\alpha = p$, which completes the proof.

6. (i) The identity can be proved by induction on n, using the following result:

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

(ii) Let $A_i = \{a_1, a_2, \ldots, a_r\}$ where $m_i = a_1 < a_2 < \ldots < a_r$. Then $n \geq a_r \geq a_{r-1} + 1 \geq \ldots \geq a_1 + (r-1) = m_i + (r-1)$, which implies that $m_i \leq n-r+1$. The fact that $1 \leq m_i$ is trivial.

(iii) For each m with $1 \leq m \leq n - r + 1$, the number of r-element subsets of S containing m as the smallest number is $\binom{n-m}{r-1}$ Thus

$$m_1 + m_2 + \ldots + m_{\binom{n}{r}} = \sum_{m=1}^{n-r+1} m\binom{n-m}{r-1}$$

(iv) By applying the identity in (i) repeatedly, we have

$$m_1 + m_2 + \ldots + m_{\binom{n}{r}} = \sum_{m=1}^{n-r+1} m\binom{n-m}{r-1}$$

$$= \binom{n}{r} + \binom{n-1}{r} + \ldots + \binom{r}{r}$$

$$= \binom{n+1}{r+1}.$$

Thus the arithmetic mean of $m_1, m_2, \ldots, m_{\binom{n}{r}}$ is

$$\binom{n+1}{r+1} / \binom{n}{r} = (n+1)/(r+1).$$